

SOME REMARKS ABOUT ELEMENTARY DIVISOR RINGS⁽¹⁾

BY

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In this and the following paper [2], we are concerned with obtaining conditions on a commutative ring S with identity element in order that every matrix over S can be reduced to an equivalent diagonal matrix⁽²⁾. Following Kaplansky [4], we call such rings *elementary divisor rings*. A necessary condition is that S satisfy

F: *all finitely generated ideals are principal.*

It has been known for some time that if S satisfies the ascending chain condition on ideals, and has no zero-divisors, then **F** is also sufficient. Helmer [3] showed that the chain condition can be replaced by the less restrictive hypothesis that S be *adequate* (i.e., of any two elements, one has a "largest" divisor that is relatively prime to the other⁽²⁾). Kaplansky [4] generalized this further by permitting zero-divisors, provided that they are all in the (Perlis-Jacobson) radical.

By a slight modification of Kaplansky's argument, we find that the condition on zero-divisors can be replaced by the hypothesis that S be an *Hermite ring* (i.e., every matrix over S can be reduced to triangular form⁽²⁾). This is an improvement, since, in any case, it is necessary that S be an Hermite ring, while, on the other hand, it is not necessary that all zero-divisors be in the radical. In fact, we show that every *regular* commutative ring with identity is adequate. However, the condition that S be adequate is not necessary either.

We succeed in obtaining a necessary and sufficient condition that S be an elementary divisor ring. Along the way, we obtain a necessary and sufficient condition that S be an Hermite ring. In the paper that follows [2], we make constant use of these results. In particular, we construct examples of rings that satisfy **F** but are not Hermite rings, and examples of Hermite rings that are not elementary divisor rings. However, all these examples contain zero-divisors; therefore, the question as to whether there exist corresponding examples that are *integral domains* is left unsettled.

DEFINITION 1. An m by n matrix A over S admits *triangular reduction* if

Presented to the Society, December 28, 1954 under the title *Concerning adequate rings and elementary divisor rings*. II; received by the editors April 20, 1955 and, in revised form, November 14, 1955.

⁽¹⁾ The preparation of this paper was sponsored (in part) by the National Science Foundation, under grant NSF G1129.

⁽²⁾ The precise definition is given below.

there exist nonsingular⁽³⁾ matrices U, V such that $AU = [b_{i,j}]$ is triangular (i.e., $b_{i,j} = 0$ whenever $i < j$), and VA is triangular; A admits diagonal reduction if there exist nonsingular matrices P, Q such that $PAQ = [c_{i,j}]$ is diagonal (i.e., $c_{i,j} = 0$ whenever $i \neq j$), and every $c_{i,i}$ is a divisor of $c_{i+1,i+1}$ [4]⁽⁴⁾.

THEOREM 2 (KAPLANSKY). *Let S be a commutative ring with identity. If all 1 by 2 and all 2 by 1 matrices over S admit diagonal reduction, then every matrix over S admits triangular reduction; in this case, S is called an Hermite ring. If, in addition, all 2 by 2 matrices over S admit diagonal reduction, then every matrix over S admits diagonal reduction; in this case, S is called an elementary divisor ring.*

For the proof, see [4, Theorems 3.5 and 5.1].

Obviously, every elementary divisor ring is an Hermite ring. Furthermore, every Hermite ring satisfies **F** [4, p. 465].

In order to prove that a given commutative ring is an Hermite ring, it suffices, by symmetry, to show only that every 1 by 2 matrix admits diagonal reduction.

THEOREM 3. *A commutative ring S with identity is an Hermite ring if and only if it satisfies the condition*

T: *for all $a, b \in S$, there exist $a_1, b_1, d \in S$ such that $a = a_1d$, $b = b_1d$, and $(a_1, b_1) = (1)$.*

Proof. Suppose that S satisfies **T**. In order to show that S is an Hermite ring, it suffices to show that every 1 by 2 matrix $[a \ b]$ admits diagonal reduction (Theorem 2 ff.). Let a_1, b_1, d, s, t satisfy $a = a_1d$, $b = b_1d$, and $sa_1 + tb_1 = 1$. Let

$$(1) \quad Q = \begin{bmatrix} s & -b_1 \\ t & a_1 \end{bmatrix}.$$

Then Q is nonsingular, and $[a \ b]Q = [d \ 0]$.

Conversely, suppose that S is an Hermite ring. Let $a, b \in S$. By hypothesis, there exists a nonsingular matrix Q , which we denote as in (1), such that $[a \ b]Q = [d \ 0]$ for some $d \in S$. Then $ab_1 = ba_1$, and $sa + tb = d$. Since Q is nonsingular, we may assume that $sa_1 + tb_1 = 1$. Then $sa_1a + tb_1a = a$, whence $sa_1a + ta_1b = a$, i.e., $a_1d = a$. Similarly, $b_1d = b$.

The following lemma, due essentially to Kaplansky [4, §4], shows that in dealing with condition **T** relative to any specific pair a, b , it suffices to consider any particular generator of the ideal (a, b) .

LEMMA 4. *Let $a, b \in S$. If a_1, b_1, d exist as in condition **T** (whence $(a, b) = (d)$),*

⁽³⁾ By *nonsingular*, we mean that U (resp. V) has a two-sided inverse in the ring of all n by n (resp. m by m) matrices over S .

⁽⁴⁾ Kaplansky [4] does not require commutativity of S .

then for all d' with $(a, b) = (d')$, there exist a'_1, b'_1 such that $a = a'_1 d'$, $b = b'_1 d'$, and $(a'_1, b'_1) = (1)$.

Proof. Write $d = kd'$, $d' = ld$, and choose s, t such that $sa_1 + tb_1 = 1$. Define $a'_1 = klt - t + a_1k$, and $b'_1 = s - kls + b_1k$. Then $a'_1 d' = a$, $b'_1 d' = b$, and $(sl - b_1)a'_1 + (tl + a_1)b'_1 = 1$.

As a straightforward consequence of Lemma 4, we have:

COROLLARY 5. *If S satisfies condition **T**, then given a, b, c, d with $(a, b, c) = (d)$, there exist a_1, b_1, c_1 such that $a = a_1 d$, $b = b_1 d$, $c = c_1 d$, and $(a_1, b_1, c_1) = (1)$.*

THEOREM 6. *A commutative ring S with identity is an elementary divisor ring if and only if it is an Hermite ring that satisfies the condition*

D': *for all $a, b, c \in S$ with $(a, b, c) = (1)$, there exist $p, q \in S$ such that $(pa, pb + qc) = (1)$.*

*Thus, S is an elementary divisor ring if and only if it satisfies **T** and **D'**.*

Proof. We have already remarked that every elementary divisor ring is an Hermite ring. The necessity of the condition **D'** is established in the proof of [4, Theorem 5.2].

The sufficiency of the two conditions is obtained by making the following two changes in the proof of [4, Theorem 5.2]. First, delete the reference to [4, Theorem 3.2]. Second, justify the fact that $xa_1 + yb_1 + zc_1$ is a unit by referring to our Corollary 5.

DEFINITION 7 (HELMER)⁽⁵⁾. A commutative ring S with identity is said to be *adequate* if it satisfies the two conditions **F** and

A: *for every $a, b \in S$, with $a \neq 0$, there exist $a_1, d \in S$ such that (i) $a = a_1 d$, (ii) $(a_1, b) = (1)$, and (iii) for every nonunit divisor d' of d , we have $(d', b) \neq (1)$.*

If in the proof of [4, Theorem 5.3], we replace the reference to [4, Theorem 5.2] by a reference to our Theorem 6, we obtain:

THEOREM 8. *An adequate ring is an elementary divisor ring if and only if it is an Hermite ring.*

DEFINITION 9 (VON NEUMANN)⁽⁶⁾. A commutative ring S with identity is said to be *regular* if for every $a \in S$, there exists $x \in S$ such that $a^2 x = a$.

von Neumann [5] shows that in any regular ring, every principal ideal is generated by an idempotent; in fact, if $a^2 x = a$, then $e = ax$ is idempotent, and $(a) = (e)$. Furthermore, every finitely generated ideal is principal; for if $b^2 y = b$, $f = by$, and $d = e + f - ef$, then $a = ad$, $b = bd$, and $d \in (e, f) = (a, b)$,

⁽⁵⁾ Helmer's definition [3] was restricted to integral domains. More general commutative rings with identity were first investigated in this connection by Kaplansky [4].

⁽⁶⁾ In von Neumann's definition [5], it is not assumed that S be commutative. The defining condition in the general case is $axa = a$.

whence $(a, b) = (d)$. Moreover, every element is a *unit* multiple of an idempotent⁽⁷⁾:

LEMMA 10. *For any element a of a regular ring S (commutative, with identity), there exists a unit u such that $a^2u = a$ (whence $e = au$ is idempotent).*

Proof. Let x satisfy $a^2x = a$, and let z satisfy $x^2z = x$. Define $u = 1 + x - xz$. Since $axz = (a^2x)xz = a^2x = a$, we have $a^2u = a$. Now obviously, $(u, x) = (1)$. But $xu = x^2$, whence x belongs to every maximal ideal that contains u . It follows that u is a unit.

THEOREM 11. *Every regular ring S (commutative, with identity) is adequate.*

Proof. We have already remarked that S must satisfy **F**. In order to show that S satisfies **A**, consider any $a, b \in S$. By Lemma 10, we may work instead with the idempotents e, f of which a, b are unit multiples. Define $d = e + f - ef$; then, as noted above, $(d) = (e, f)$. Put $e_1 = 1 - f + ef$. Then $e = e_1d$ and $(e_1, f) = (1)$. Since d divides f , no nonunit divisor d' of d can be relatively prime to f .

REMARK 12. Kaplansky points out [4, p. 474] that by using results developed in [1], one can show that every commutative regular ring S with identity is an elementary divisor ring. This can also be seen as follows. Working again with the idempotents e and f , let d and e_1 be as above, and define $f_1 = f$. Then $e = e_1d$, $f = f_1d$, and $(e_1, f_1) = (1)$. It follows that S is an Hermite ring (Theorem 3). Therefore, by Theorem 8, S is an elementary divisor ring.

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(7) The arguments that follow are motivated by [1, Theorem 2.2].